8: quaternions and spectral theorems

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One particular special case is the relationship between unit quaternions and rotations, which is used often in practice in computer graphics.

Def 15.1 The unit quaternions are the elements of the group $SU(2)$:

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \text{ s.t. } \alpha, \beta \in \mathbb{C},\ \alpha\bar{\alpha} + \beta\bar{\beta} = 1 \right\}.$$

$$\left(\begin{bmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}^* = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}^{-1} = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix} \quad \Rightarrow \delta = \bar{\alpha},\ \bar{\gamma} = -\beta \Rightarrow \gamma = -\bar{\beta}. \right)$$

The quaternions are the elements of the real vector space $\mathbb{H} = \mathbb{R}\, SU(2)$.

Let $\underset{\sim}{1}, \underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ be the matrices

$$\underset{\sim}{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \underset{\sim}{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ \underset{\sim}{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \underset{\sim}{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Then $\mathbb{H} = \{ X = a\underset{\sim}{1} + b\underset{\sim}{i} + c\underset{\sim}{j} + d\underset{\sim}{k},\quad a, b, c, d \in \mathbb{R} \}$.

Notation: While working in the quaternions, let's drop the $\underset{\sim}{[\,]}$ below $1, i, j, k$

Can verify
$$i^2 = j^2 = k^2 = ijk = -1$$
$$ij = -ji = k$$
$$jk = -kj = i$$
$$ki = -ik = j.$$

Quaternions are a generalization of the complex numbers, and turn out to be a skew field (non commutative field).

$\mathbb{H}$ is isomorphic to $\mathbb{R}^4$ as a vector space.

Last time I stated that we can use quaternions to represent rotations

Last time I stated that we can use quaternions to represent rotations in $SO(3)$, which turns out to have all sorts of applied applications in robotics, computer vision, and graphics

Def. 15.2 A concise notation for the quaternion $X = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ defined by $\alpha = a + ib$ and $\beta = c + id$ is

$$X = [a, (b, c, d)], \text{ where}$$

a is the scalar part of X and (b, c, d) is the vector part of X.

Note that $X^* = [a, -(b, c, d)]$, which we will also denote $\bar{X}$, the conjugate of X.

If X is a unit quaternion, then $\bar{X}$ is the multiplicative inverse of X.

(Lie algebra satisfying Lie bracket)

Def. 15.3 The real vector space $\mathfrak{su}(2)$ of 2×2 skew Hermitian $\to A^* = -A$ matrices with 0 trace is given by

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} ix & y + iz \\ -y + iz & -ix \end{pmatrix} \,\middle|\, (x, y, z) \in \mathbb{R}^3 \right\}$$

Def. 15.4 The adjoint representation of the group $SU(2)$ is the group homomorphism $Ad : SU(2) \to GL(\mathfrak{su}(2))$ defined s.t. $\forall q \in SU(2)$, with $q = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$, we have $Ad_q(A) = qAq^*$, $A \in \mathfrak{su}(2)$

where q^* is the inverse of q, $q^* = \begin{pmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{pmatrix}$, $qq^* = I$.

Need to verify $Ad_q : \mathfrak{su}(2) \to \mathfrak{su}(2)$ is invertible and Ad is a group homomorphism.

Then we can embed $\mathbb{R}^3$ into $\mathbb{H}$ by (we care about only "pure" quaternions)

$$\Psi(x, y, z) = \begin{pmatrix} ix & y + iz \\ -y + iz & -ix \end{pmatrix}.$$

Then q defines the map $\rho_q : \mathbb{R}^3 \to \mathbb{R}^3$ by

Then q defines the map $\rho_q : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by

$$\rho_q(x,y,z) = \Psi^{-1}(q\Psi(x,y,z)q^*).$$

It turns out this ρ_q is a rotation, and we can represent rotations in $SO(3)$ by the adjoint representation of $SU(2)$.

Recall that we can embed $\mathbb{R}^3$ in $\mathbb{H}$ by

$$\Psi(x,y,z) = \begin{pmatrix} ix & y+iz \\ -y+iz & -ix \end{pmatrix}, \quad x,y,z \in \mathbb{R}.$$

Technically, it's a little more convenient to embed $\mathbb{R}^3$ in the real vector space of Hermitian matrices with 0 trace

$$\left\{ \begin{pmatrix} x & z-iy \\ z+iy & -x \end{pmatrix}, \ x,y,z \in \mathbb{R} \right\}.$$

Then $-i\Psi(x,y,z) = \begin{pmatrix} x & z-iy \\ z+iy & -x \end{pmatrix} = x\sigma_3 + y\sigma_2 + z\sigma_1$,

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

are called the Pauli spin matrices

Note also that $(\mathbf{i} = i\sigma_3, \ \mathbf{j} = i\sigma_2, \ \mathbf{k} = i\sigma_1)$ form a basis of $\mathfrak{su}(2)$.

If $A = x\sigma_3 + y\sigma_2 + z\sigma_1$ is a Hermitian matrix with 0 trace,

$$(qAq^*)^* = qA^*q^* = qAq^*, \text{ so } qAq^* \text{ is Hermitian} \qquad (q \in SU(2))$$

and $tr(qAq^*) = tr(Aq^*q) = tr(A) = 0$.

So the map $A \mapsto qAq^*$ is still a Hermitian matrix with 0 trace.

Further, $\det(x\sigma_3 + y\sigma_2 + z\sigma_1) = \det\begin{pmatrix} x & z-iy \\ z+iy & -x \end{pmatrix} = -(x^2+y^2+z^2) = \det(qAq^*)$

So we can use the mapping

$$\varphi(x,y,z) = x\sigma_3 + y\sigma_2 + z\sigma_1 \text{ from } \mathbb{R}^3 \text{ to } 0\text{-trace Hermitian matrices.}$$

Note: $\varphi = -i\Psi$ and $\varphi^{-1} = i\Psi^{-1}$

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Def. 15.6 The unit quaternion $q \in SU(2)$ induces a map r_q on $\mathbb{R}^3$ by $r_q(x, y, z) = \varphi^{-1}(q\varphi(x, y, z)q^*) = \varphi^{-1}(q(x\sigma_3 + y\sigma_2 + z\sigma_1)q^*)$.

r_q is clearly linear because φ is linear.

Prop. 15.1 For every unit quaternion $q \in SU(2)$, the linear map r_q is orthogonal, i.e. $r_q \in O(3)$.

proof. $-\|(x, y, z)\|^2 = -(x^2 + y^2 + z^2) = \det(\varphi(x, y, z))$

$\Rightarrow -\|r_q(x, y, z)\|^2 = \det(\varphi(r_q(x, y, z))) = \det(q(x\sigma_3 + y\sigma_2 + z\sigma_1)q^*)$

$= \det(x\sigma_3 + y\sigma_2 + z\sigma_1) = -\|(x, y, z)\|^2.$

$\Rightarrow r_q \in O(3).$

Prop. 15.2 Let $q = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$, $\alpha = a + bi$, $\beta = c + di$. If $(b, c, d) \neq (0, 0, 0)$, then the fixed pts of r_q are the solutions (x, y, z) of the linear system

$$\begin{aligned} -dy + cz &= 0 \\ cx - by &= 0 \\ dx - bz &= 0. \end{aligned}$$

This linear system has the nontrivial solution (b, c, d) and rank 2. Therefore, r_q has the eigenvalue 1 with multiplicity 1, and r_q is a rotation whose axis is determined by (b, c, d).

proof sketch Lots of symbol manipulation to get to the linear system.

Rewrite as $\begin{pmatrix} 0 & -d & c \\ c & -b & 0 \\ d & 0 & -b \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$

Note $(b, c, d) \neq (0, 0, 0)$ is a soln, so rank < 3.

But no matter the values of b, c, d, there is at least one nonsingular 2×2 submatrix, so rank ≥ 2 $\Rightarrow$ rank $= 2$.

$\Rightarrow r$ has eigenvalue 1 with multiplicity 1.

2×2 submatrix, so rank $= 2$ ...

$\Rightarrow r_q$ has eigenvalue 1 with multiplicity 1.

Suppose $\det(r_q) = -1$. Then eigenvalues are either

$(-1, 1, 1)$ or $(-1, e^{i\theta}, e^{-i\theta})$ with $\theta \neq k2\pi$, $k \in \mathbb{Z}$,

a contradiction.

$\Rightarrow \det(r_q) = 1$. $\Rightarrow r_q$ is a rotation if $(b,c,d) \neq (0,0,0)$.

Aside: If $(b,c,d) = (0,0,0)$, then r_q is the identity.

Thm 15.1/15.3 The map $r: SU(2) \to SO(3)$ is a homomorphism whose kernel is $\{I, -I\}$.

proof. Let $q_1, q_2 \in SU(2)$.

Then
$$
\begin{aligned}
r_{q_2}(r_{q_1}(x,y,z)) &= \varphi^{-1}(q_2\, \varphi(r_{q_1}(x,y,z))\, q_2^*) \\
&= \varphi^{-1}(q_2\, \varphi(\varphi^{-1}(q_1 \varphi(x,y,z) q_1^*))\, q_2^*) \\
&= \varphi^{-1}(q_2 q_1\, \varphi(x,y,z)\, q_1^* q_2^*) \\
&= r_{q_2 q_1}(x,y,z).
\end{aligned}
$$

$\Rightarrow r$ is a homomorphism.

Recall that if $r_q = I_3$, then $(b,c,d) = (0,0,0)$.

But then $a = \pm 1$, $\Rightarrow q = \pm I_2$.

$\Rightarrow \operatorname{Ker}(r) = \{I_2, -I_2\}$.

So we now have a mapping from $SU(2) \to SO(3)$.

Prop. 15.3/15.4 The matrix representing r_q is

$$
R_q = \begin{pmatrix} 2a^2+2b^2-1 & 2bc-2ad & 2ac+2bd \\ 2bc+2ad & 2a^2+2c^2-1 & -2ab+2cd \\ -2ac+2bd & 2ab+2cd & 2a^2+2d^2-1 \end{pmatrix}
$$

proof. Lots of tedious computation

Thm 15.2/15.4 The homomorphism $r: SU(2) \to SO(3)$ is surjective.

proof sketch: The book gives an explicit algorithm for finding a unit quaternion representing a rotation matrix. $\square$

It turns out that computing on rotations using quaternions is often much more efficient than doing it on rotation matrices, so this is used very often in practice, but we won't go into more detail here.

Spectral theorems in Euclidean and Hermitian spaces

Def. 16.1 Given a Euclidean or Hermitian space E, a linear map $f: E \to E$ is

- normal if $f \circ f^* = f^* \circ f$
- self-adjoint if $f = f^*$
- skew self-adjoint if $f = -f^*$
- orthogonal if $f \circ f^* = f^* \circ f = \mathrm{Id}$

Prop. 16.1 Given a Euclidean or Hermitian space E, if $f: E \to E$ is a normal linear map, then $\mathrm{Ker}\, f = \mathrm{Ker}\, f^*$.

proof.

$$\begin{aligned} \langle f(u), f(v) \rangle &= \langle u, (f^* \circ f)(v) \rangle \\ &= \langle u, (f \circ f^*)(v) \rangle \\ &= \langle f^*(u), f^*(v) \rangle \quad \forall u, v \in E. \end{aligned}$$

Since $\langle , \rangle$ is pos. def.,

$$\langle f(u), f(u) \rangle = 0 \quad \text{iff} \quad f(u) = 0$$

$$\langle f^*(u), f^*(u) \rangle = 0 \quad \text{iff} \quad f^*(u) = 0.$$

$$\Rightarrow f(u) = 0 \quad \text{iff} \quad f^*(u) = 0$$

$$\Rightarrow \mathrm{Ker}\, f = \mathrm{Ker}\, f^*.$$

Prop. 16.2 Given a Hermitian space E, for any normal linear map

Prop. 16.2 Given a Hermitian space E, for any normal linear map $f: E \to E$, $\ker f \cap \operatorname{Im} f = (0)$.

proof. Let $v \in \ker f \cap \operatorname{Im} f$. Then $v = f(u)$ for some $u \in E$ and $f(v) = 0$.
By prev Prop, $f^*(v) = 0$. $\Rightarrow$ $0 = \langle f^*(v), u \rangle = \langle v, f(u) \rangle = \langle v, v \rangle$
$\Rightarrow$ $v = 0$. ☐

So repeated applications of normal linear maps don't send more things to 0.

Prop. 16.3 Given a Hermitian space E, for any normal linear map $f: E \to E$, a vector u is an eigenvector of f for the eigenvalue $\lambda \in \mathbb{C}$ iff u is an eigenvector of f^* for the eigenvalue $\bar{\lambda}$.

proof. Note that $(f - \lambda id)$ is normal (can verify directly).
Then $(f - \lambda id)u = 0$ iff $(f^* - \bar{\lambda} id)(u) = 0$. ☐

Prop. 16.4 If u, v are eigenvectors of a normal linear map $f: E \to E$, (E Hermitian) associated with distinct eigenvalues $\lambda \neq \mu$, then $\langle u, v \rangle = 0$.

proof. $\lambda \langle u, v \rangle = \langle \lambda u, v \rangle = \langle f(u), v \rangle = \langle u, f^*(v) \rangle = \langle u, \bar{\mu} v \rangle = \mu \langle u, v \rangle$.
$\Rightarrow$ $\langle u, v \rangle = 0$. ☐

Prop. 16.5 Given a Hermitian space E, all eigenvalues of a self-adjoint linear map $f: E \to E$ are real.

proof. Let (z, u) be an eigenvalue/vector pair for f.
Then $z \langle u, u \rangle = \langle f(u), u \rangle = \langle u, f^*(u) \rangle = \langle u, f(u) \rangle = \langle u, zu \rangle = \bar{z} \langle u, u \rangle$
$\Rightarrow$ $z = \bar{z}$. ☐

Def. 16.2 Given a real vector space E, let $E_{\mathbb{C}}$ be the structure $E \times E$ with addition $(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$
and multiplication $(x + iy) \cdot (u, v) = (xu - yv, yu + xv)$,
a complex vector space. $(u, v) = u + iv$.

We can extend $f: E \to E$ to $f_{\mathbb{C}}: E_{\mathbb{C}} \to E_{\mathbb{C}}$ by $f_{\mathbb{C}}(u+iv) = f(u) + i f(v)$.

Note for any basis $(e_1, \dots, e_n)$ of E, $M(f) = M(f_{\mathbb{C}})$.

Given a Euclidean inner product $\langle\ ,\ \rangle$ on E, we can get a Hermitian inner product $\langle u_1 + iv_1, u_2 + iv_2 \rangle_{\mathbb{C}} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle + i(\langle v_1, u_2 \rangle - \langle u_1, v_2 \rangle)$

Further, $f_{\mathbb{C}}^*(u+iv) = f^*(u) + i f^*(v)$ is the adjoint of $f_{\mathbb{C}}$ w.r.t. $\langle\ ,\ \rangle_{\mathbb{C}}$.

Prop. 16.6 Given a Euclidean space E, if $f: E \to E$ is any self-adjoint linear map, then every eigenvalue λ of $f_{\mathbb{C}}$ is real and is an eigenvalue of f, so all eigenvalues of f are real.

proof sketch Look at characteristic polynomials $\square$

Prop. 16.7 Given a Hermitian space E, for any linear map $f: E \to E$,
if f is skew self-adjoint, then its eigenvalues are pure imaginary or 0,
if f is unitary, then f has eigenvalues of absolute value 1.

proof. If $f^* = -f$, then for (λ, u) an eigenpair of f,

$$\lambda \langle u, u \rangle = \langle f(u), u \rangle = \langle u, f^*(u) \rangle = \langle u, -f(u) \rangle = -\bar{\lambda} \langle u, u \rangle$$

$$\Rightarrow \lambda = -\bar{\lambda} \quad \Rightarrow \lambda = ir, \ r \in \mathbb{R}.$$

If f is unitary, then f is an isometry, so for any eigenpair (λ, u),

$$\langle u, u \rangle = \langle f(u), f(u) \rangle = \langle \lambda u, \lambda u \rangle = \lambda \bar{\lambda} \langle u, u \rangle = |\lambda|^2 \langle u, u \rangle.$$

$$\Rightarrow |\lambda| = 1. \quad \square$$

Thm 16.1/16.8 (Spectral Thm for self-adjoint linear maps on Euclidean space)

Given a Euclidean space E of dim n, for every self-adjoint linear map $f: E \to E$, there is an orthonormal basis $(e_1, \dots, e_n)$ of eigenvectors of f s.t. $M(f)$ is a diagonal matrix

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \quad \lambda_i \in \mathbb{R}.$$

proof. By induction on dimension n. If $n = 1$, trivial.
For $n \geq 2$, pick an eigenvalue $\lambda \in \mathbb{R}$, with a unit eigenvector w.
Let $W = \mathbb{R} w$. Then $f(W) \subseteq W$.

For $n \geq 2$, pick an eigenvalue $\lambda \in \mathbb{R}$, with

Let $W = \mathbb{R} w$. Then $f(W) \subseteq W$.

Furthermore $f(W^{\perp}) \subseteq W^{\perp}$ because $\forall\, v \in W^{\perp}$,

$$\begin{aligned} \langle f(v), w \rangle &= \langle v, f(w) \rangle \\ &= \langle v, \lambda w \rangle \\ &= \lambda \langle v, w \rangle = 0 \quad \text{since } v \in W^{\perp}. \end{aligned}$$

Note $\dim(W^{\perp}) = n-1$, and f restricted to $W^{\perp}$ is also self-adjoint, so we can proceed with induction.

Prop. 16.8/16.9 Given a Hermitian space E, for any linear map $f: E \to E$, and any subspace W of E, if $f(W) \subseteq W$, then $f^*(W^{\perp}) \subseteq W^{\perp}$. Thus if $f(W) \subseteq W$ and $f^*(W) \subseteq W$, then $f(W^{\perp}) \subseteq W^{\perp}$ and $f^*(W^{\perp}) \subseteq W^{\perp}$.

proof. Let $u \in W^{\perp}$. Then $\langle w, u \rangle = 0$ $\forall\, w \in W$.

But $\langle f(w), u \rangle = \langle w, f^*(u) \rangle$, so if $f(W) \subseteq W$, $LHS = 0$, so $f^*(u) \in W^{\perp}$

$\Rightarrow f^*(W^{\perp}) \subseteq W^{\perp}$. $\square$

Prop. 16.9/16.10 If $f: E \to E$ (Euclidean E) is a linear map and $w = u + iv$ is an eigenvector of $f_{\mathbb{C}}: E_{\mathbb{C}} \to E_{\mathbb{C}}$ for the eigenvalue $z = \lambda + i\mu$, where $u, v \in E$ and $\lambda, \mu \in \mathbb{R}$, then

$$f(u) = \lambda u - \mu v \quad \text{and} \quad f(v) = \mu u + \lambda v.$$

Thus,

$$f_{\mathbb{C}}(u - iv) = f(u) - i f(v) = (\lambda - i\mu)(u - iv),$$

so $\bar{w} = u - iv$ is an eigenvector of $f_{\mathbb{C}}$ for $\bar{z} = \lambda - i\mu$.

proof. $f_{\mathbb{C}}(u + iv) = (\lambda + i\mu)(u + iv) = \lambda u - \mu v + i(\mu u + \lambda v)$

$f_{\mathbb{C}}(u + iv) = f(u) + i f(v)$ $\Rightarrow$ $f(u) = \lambda u - \mu v$ and $f(v) = \mu u + \lambda v$. $\square$

Prop. 16.10/16.11 Given a Euclidean space E, for any normal linear map $f: E \to E$, if $(z = \lambda + i\mu,\ w = u + iv)$ is an eigenpair of $f_{\mathbb{C}}$, then $\langle u, v \rangle = 0$ and $\langle u, u \rangle = \langle v, v \rangle$, and if $W = \operatorname{span}\{u, v\}$, then $f(W) = W$ and $f^*(W) = W$. Furthermore, w.r.t. the orthogonal basis (u, v), $M(f_W) = \begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix}$.

If $\mu = 0$, then λ is a real eigenvalue of f with two eigenvectors u and v.
If $W = \operatorname{span}(u)$, $u \neq 0$, or $W = \operatorname{span}(v)$, $u = 0$, then $f(W) \subseteq W$ and $f^*(W) \subseteq W$.

proof. Note $(\bar{z} = \lambda - i\mu,\ \bar{w} = u - iv)$ is an eigenpair of $f_{\mathbb{C}}$.
If $\mu \neq 0$, then this pair is distinct and

$$0 = \langle w, \bar{w} \rangle = \langle u, u \rangle - \langle v, v \rangle + 2i\langle u, v \rangle$$

$\Rightarrow \langle u, v \rangle = 0$ and $\langle u, u \rangle = \langle v, v \rangle$. (Note $w \neq 0$, so $u \neq 0$, $v \neq 0$)

$\Rightarrow u, v$ are orthogonal and lin. ind.

But $f(u) = \lambda u - \mu v$ and $f(v) = \mu u + \lambda v$, so $f(W) = W$
$f^*(u) = \lambda u + \mu v$ and $f^*(v) = -\mu u + \lambda v$, so $f^*(W) = W$. $\}$ if $W = \operatorname{span}(u, v)$.

$\square$

(Main spectral thm)

Thm 16.2/16.12 Given a Euclidean space E of dim n, for every normal linear map $f: E \to E$, there is an orthonormal basis $(e_1, \dots, e_n)$ s.t.

$$M(f) = \begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_p \end{bmatrix},$$

where A_j is either 1×1 or $A_j = \begin{pmatrix} \lambda_j & -\mu_j \\ \mu_j & \lambda_j \end{pmatrix}$, $\mu_j > 0$.

proof sketch: Induction on dim n, keep building up basis.

Thm 16.3/16.13 Given a Hermitian space of dim n, for every normal linear map $f: E \to E$, there is an orthonormal basis $(e_1, \dots, e_n)$ of eigenvectors s.t.

$$M(f) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \quad \lambda_i \in \mathbb{C}.$$

of eigenvectors s.t. $M(f) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, $\lambda_i \in \mathbb{C}$.